B.sc(H) part 2

paper 3

Topic:Uniqueness of identity elements &Inverse element in a group

UG

Sbject:Mahtematics

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1:cancellation laws in a group

Theorem

. If $a, b, c \in G$, then

(i) $ab = ac \Rightarrow b = c$ (left cancellation law)

(ii) $ba = ca \Rightarrow b = c$ (right cancellation law)

Proof : (i) Given that $ab = ac$. ... (I)

Let a^{-1} be the inverse of a in G. Multiplying (i.e. applying the group operation) both sides of (1) by a^{-1} on the left, we get.

$$a^{-1}(ab) = a^{-1}(ac)$$

which by associative law becomes

$$(a^{-1}a)b = (a^{-1}a)c.$$

Since by postulate (G_4), $a^{-1}a = e$, the identity in G, we have

$$eb = ec.$$

Now by postulate (G_3), we have $eb = b$ and $ec = c$.

Therefore we get $b = c$ and the first part of the theorem is proved.

(ii) Given that $ba = ca$. ... (II)

Let a^{-1} be the inverse of a in G. Multiplying both sides of II by a^{-1} on the right, we get

$$(ba)a^{-1} = (ca)a^{-1}$$

$\Rightarrow \quad b(aa^{-1}) = c(aa^{-1})$ [by postulate G_2]

$\Rightarrow \quad be = ce$ [by postulate G_4]

$\therefore \quad b = c$ [by postulate G_3]

Theorem
The identity element in a group is unique

Proof : Let G be a group and let e be an identity element.

We have to prove that e is unique.

If not, suppose e' be another identity element in a group G.

Since e is the identity element of G, therefore.

$$ae = ea = a \quad \text{.. (1)}$$

Similarly since e' is the identity element of G, therefore

$$ae' = e'a = a \quad \dots (2)$$

for every $a \in G$.

Since the equation (1) is true for every $a \in G$ and since $e' \in G$, therefore putting $a = e'$ in (1) we get

$$e'e = ee' = e' \quad \dots (3)$$

Similarly putting $a = e$ in (2), we get

$$ee' = e'e = e \quad \dots (4)$$

Hence from (3) and (4), it follows that $e = e'$ which means that the identity in a group is unique.

Second Method : From (1) and (2), we have $ae = ae'$. Therefore from the cancellation law $e = e'$.

Hence the identity in a group is unique.

Teorem
The inverse of an element in a group is unique

Proof : Let G be a group Let a be an element of G and let a^{-1} be its inverse.

We have to prove that a^{-1} is unique. If not, suppose a' is another inverse of a.

Since a^{-1} is the inverse of a, therefore

$$aa^{-1} = a^{-1}a = e \quad \dots (1)$$

Similarly since a' is the inverse of a, therefore

$$aa' = a'a = e \quad \dots (2)$$

where e is the identity element of G.

Multiplying (1) by a' on the left, we get

$$a'(aa^{-1}) = a'e = a' \quad \dots (3)$$

Multiplying (2) by a^{-1} on the right, we get

$$(a'a)a^{-1} = ea^{-1} = a^{-1} \quad \dots (4)$$

But by associative law

$$a'(aa^{-1}) = (a'a)a^{-1}$$

Therefore we have from (3) and (4),

$$a' = a^{-1}$$

Hence the inverse of an element in a group is unique.

Second Method : From (1) and (2) we have $aa' = aa^{-1}$. Therefore from the cancellation law $a' = a^{-1}$.

Hence the inverse in a group is unique.